FINITE-TIME BLOW-UP PROBLEM AND THE MAXIMUM GROWTH OF PALINSTROPHY

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Abstract This investigation is a part of a broader research effort aiming to discover solutions of the Navier-Stokes system in 2D and 3D which can saturate certain analytically obtained bounds on the maximum growth of enstrophy and palinstrophy [4, 5]. This research is motivated by questions concerning the possibility of finite-time blow-up of solutions of the 3D Navier-Stokes system where such estimates play a key role. We argue that insights concerning the sharpness of these estimates can be obtained from the numerical solution of suitably-defined PDE optimization problems. In the present contribution we focus on the sharpness of the analytical bounds on the instantaneous rate of growth of palinstrophy $\mathcal{P}$ in 2D incompressible flows and identify the vortex configurations which maximize this quantity under certain constraints. These optimal vortex states exhibit a distinct scale-invariant structure and offer insights about mechanisms leading to generation of small scales in the enstrophy cascade.

INTRODUCTION

The research program outlined here is motivated by the question concerning existence for arbitrarily large times of smooth solutions to the 3D Navier-Stokes system corresponding to smooth initial data of arbitrary size. To date, smooth solutions have been proved to exist for *finite* times only [1, 2], leaving open the possibility of a spontaneous formation of singularities in finite times. We mean by this the loss of regularity of the solution manifested by the "blow-up" of its certain norms. The importance of this question has been recognized by the Clay Mathematics Institute which identified it as one of the "millennium problems" [3]. There are also analogous questions concerning the existence of smooth solutions to the 3D Euler equations. It is believed that, should such blow-up occur in finite time, it is associated with the formation and amplification of the small-scale vortex structures. Indeed, a number of vortex flows have been proposed, e.g., [6], which might possibly lead to singularities in finite time, although the computational evidence is not conclusive. All of these candidate flows were postulated in a rather ad-hoc fashion based on purely physical arguments. The goal of the present research effort is to perform search for such singular vortex structures in a systematic manner using rigorous methods of mathematical optimization.

The key observation is that the question about finite-time singularity formation can be cast in terms of boundedness of certain norms of the solution. More precisely, if $\mathbf{u}$ is a 3D velocity vector field, then it is well-known that the boundedness of the *enstrophy* $\mathcal{E}(t) := \frac{1}{2}\int_\Omega |\boldsymbol{\nabla} \times \mathbf{u}(t,\mathbf{x})|^2 \, d\Omega$ will guarantee smoothness of the solution up to time $t > 0$ [2]. Using methods of functional analysis, the rate of growth of enstrophy can be estimated as $d\mathcal{E}(t)/dt < C\mathcal{E}(t)^3$ which, upon integration with respect to time, leads to $\mathcal{E}(t) \le \mathcal{E}(0)/\sqrt{1 - 4tC\mathcal{E}(0)^2/\nu^3}$. We note that this upper bound, which is the sharpest result of this kind available to date [2], blows up in finite time, hence based on this estimate, singularity formation cannot be ruled out. Thus, the question about the possibility of finite-time blow-up can be framed in terms of *sharpness* of estimate $d\mathcal{E}(t)/dt < C\mathcal{E}(t)^3$ (by "sharpness" we mean existence of a solution which *saturates* a given inequality upper bound). Analogous questions can also be studied in the case of simpler systems, such as the 1D Burgers and 2D Navier-Stokes equations, where they lead to computational problems which are more tractable. While it is known that both 1D Burgers and 2D Navier-Stokes systems have solutions which are smooth for arbitrary times, for such systems as well there exist analytical upper bounds for the growth of various quadratic quantities, and since they are obtained with similar methods as for the 3D Navier-Stokes system, questions about their sharpness are in fact quite relevant. Best results currently available as regards the sharpness of a number of important estimates in 1D, 2D and 3D, both instantaneous and finite-time, are collected in Table 1. Addressing the outstanding questions indicated in the rightmost column of the Table constitutes a long-term goal of the present research effort.

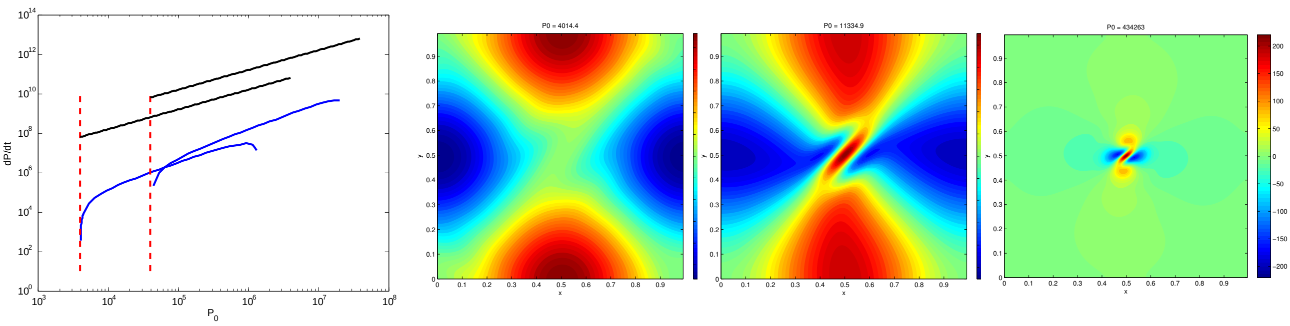
SEARCH FOR EXTREMAL SOLUTIONS

Sharpness of estimates listed in Table 1 can be probed using variational optimization methods which allow one to find flows leading to the largest possible growth of a given quantity. Lu & Doering [4] used this approach to demonstrate that the 3D instantaneous estimate $d\mathcal{E}(t)/dt < C\mathcal{E}(t)^3$ is in fact sharp, whereas the present authors used it to study the finite-time estimates for the 1D Burgers problem [5]. Here we review some recent results concerning the most singular vortex structures in 2D Navier-Stokes flows. Since in such flows in doubly-periodic domains the enstrophy can only decrease ($d\mathcal{E}(t)/dt < 0$), the relevant quantity is the *palinstrophy* $\mathcal{P}(\omega(t)) := \frac{1}{2}\int_\Omega |\boldsymbol{\nabla}\omega(t,\mathbf{x})|^2 \, d\Omega$, where $\omega := \partial_x v - \partial_y u$ is the vorticity component perpendicular to the plane of motion. Palinstrophy is associated with the equation for the evolution of the vorticity gradient $\boldsymbol{\nabla}\omega$ which features a "stretching term" similar to the one in the 3D vorticity equation. The rate of growth of palinstrophy can be upper-bounded as [7]

$$d\mathcal{P}(t)/dt \le C\,\mathcal{E}\,\mathcal{P}, \qquad C > 0 \tag{1}$$

	Best Estimate	Sharpness
1D Burgers instantaneous	$\frac{dE(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3} E(t)^{5/3}$	Yes Lu & Doering [4]
1D Burgers finite-time	$\max_{t\in[0,T]} E(t) \leq \left[E_0^{1/3} + \left(\frac{L}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} E_0\right]^3$	No Ayala & Protas [5]
2D Navier-Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq \frac{C}{\nu}\mathcal{E}\,\mathcal{P}$	present work
2D Navier-Stokes finite-time	$\max_{t>0} \mathcal{P}(t) \leq \mathcal{P}(0) + \frac{C_1}{2\nu^2}\frac{L^4}{16\pi^4}\mathcal{P}(0)^2$	?
3D Navier-Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering [4]
3D Navier-Stokes finite-time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	???

Table 1. Summary of the best estimates available to date for the instantaneous rate of growth and the growth over finite time of enstrophy and palinstrophy in 1D Burgers, 2D and 3D Navier-Stokes systems.



(a) (b) (c) (d)

Figure 1. (a) (blue) Maximum rate of growth $d\mathcal{P}/dt$ versus $\mathcal{P}_0$ obtained from the numerical solution of problem (2) and (black) analytical estimate (1); (b)–(d) corresponding maximizing vorticity fields with $\mathcal{E}_0 = 100$ and $\mathcal{P}_0 = 4.0 \times 10^3, 1.1 \times 10^4, 4.3 \times 10^5$.

and the question about the 2D vortex structures saturating this growth of palinstrophy at a fixed time leads to the following optimization problem

$$\max_{\omega \in H^2(\Omega)} d\mathcal{P}(\omega)/dt \ \text{ subject to } \ \mathcal{E}(\omega) = \mathcal{E}_0,\ \mathcal{P}(\omega) = \mathcal{P}_0, \tag{2}$$

where $\mathcal{E}_0$ and $\mathcal{P}_0$ are prescribed reference enstrophy and palinstrophy levels. Solution of optimization problem (2) can be obtained using variational methods of PDE optimization and details are described in [8]. Figure 1a shows the maximum instantaneous growth of palinstrophy $d\mathcal{P}/dt$ obtained for a broad range of palinstrophy values $\mathcal{P}_0$ with enstrophy fixed at $\mathcal{E}_0 = 100$ and $\mathcal{E}_0 = 1000$. Due to Poincaré's inequality $\mathcal{E}_0 \leq (2\pi)^{-2}\mathcal{P}_0$ bounding the smallest palinstrophy values from below, the two branches in Figure 1a exist only for sufficiently large values of $\mathcal{P}_0$. The growth rate $\max d\mathcal{P}/dt$ vs. $\mathcal{P}_0$ observed in Figure 1a indicates that estimate (1) is in fact sharp. The corresponding vortex states which saturate this estimate are shown in Figures 1b-c. Interestingly, they reveal a scale-invariant structure emerging as $\mathcal{P}_0 \to \mathcal{P}_0^{\max}$, where $\mathcal{P}_0^{\max}$ is the largest palinstrophy value for which maximizing solutions can be found. Since the identified vortex states maximize the instantaneous rate of palinstrophy production, we will also discuss their relevance for understanding the mechanisms underlying the enstrophy cascade in 2D turbulence.

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