EXACT KOLMOGOROV LAW FOR COMPRESSIBLE TURBULENCE

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Abstract Compressible turbulence is still a subject poorly understood despite the number of applications ranging from aeronautics to astrophysics. In order to better understand recent direct numerical simulations made in the context of interstellar turbulence which reveal intermittency with anomalous scalings we have investigated isothermal hydrodynamics under the assumption of homogeneity and in the asymptotic limit of a high Reynolds number. An exact relation has been derived for some two-point correlation functions which reveals a fundamental difference with the incompressible case for which we have the classical Kolmogorov ($4/5$ or $4/3$) law. The main difference resides in the presence of a new type of term which acts on the inertial range similarly as a source or a sink for the mean energy transfer rate. When isotropy is assumed, compressible turbulence may be described by the relation [4]:

$$-\frac{2}{3}\varepsilon_{\rm eff} r = F_r(r)\,, \tag{1}$$

where F_r is the radial component of a two-point correlation functions and $\varepsilon_{\rm eff}$ is an effective mean total energy injection rate. By dimensional arguments, we predict that a spectrum in $k^{-5/3}$ may still be preserved at small scales if the density-weighted fluid velocity $\rho^{1/3}u$ is used. A steeper power law is expected at the largest scales if the effective mean total energy exhibits a scale dependence. The theoretical predictions are in relatively good agreement with the most recent direct numerical simulations.

INTRODUCTION

Although significant advances have been made in the regime of wave turbulence for which a systematic analysis is possible [9], the regime of strong turbulence continues to resist modern efforts at solution; for that reason any exact result is of great importance. In his third 1941 turbulence paper Kolmogorov derived an exact relation for incompressible isotropic hydrodynamics in terms of third-order structure function and in the asymptotic limit of a high Reynolds number (Re) [6]. Because of the rarity of such results, the Kolmogorov's four-fifths law has a cornerstone role in the analysis of turbulence. Few extensions of such results to other incompressible fluids have been made; it concerns for example scalar passively advected such as the temperature or a pollutant in the atmosphere, quasi-geostrophic flows or astrophysical magnetized fluids described in the framework of Hall MHD [3].

Our knowledge of compressible hydrodynamic turbulence is mainly limited to direct numerical simulations. The most recent results for supersonic isothermal turbulence with a grid resolution up to 2048^3 [2, 8] reveal that the inertial range velocity scaling deviates substantially from the incompressible Kolmogorov spectrum with a slope of the velocity power spectrum close to -2 and an exponent of the third-order velocity structure function of about 1.3. Surprisingly, the incompressible predictions are shown to be restored if the density-weighted fluid velocity, $\rho^{1/3}\mathbf{u}$, is used instead of simply the velocity $\mathbf{u}$. Although a -2 spectrum may be associated with shocks – like in one dimension – it seems that their contribution in three dimensions is more subtle. Generally speaking it is fundamental to establish the equivalent of the $4/5$s law for compressible turbulence before going to the more difficult problem of intermittency.

COMPRESSIBLE ISOTHERMAL HYDRODYNAMIC TURBULENCE

In this presentation, compressible isothermal hydrodynamic turbulence will be discussed in the limit of high Re. We shall present the nature of such a compressible turbulence through an analysis in the physical space in terms of two-point correlation functions. In particular, the discussion will be focused on the isotropic case for which a simple exact relation emerges. Our result will be put in the general context of observations of interstellar turbulence [5] and of direct numerical simulations where important results have been obtained recently.

The exact relation that we found for an homogeneous compressible turbulence may be written as [4]:

$$-2\varepsilon = \langle(\nabla'\cdot\mathbf{u}')(R-E)\rangle + \langle(\nabla\cdot\mathbf{u})(\tilde{R}-E')\rangle + \nabla_{\mathbf{r}}\cdot\left\langle\left[\frac{\delta(\rho\mathbf{u})\cdot\delta\mathbf{u}}{2} + \delta\rho\delta e - C_s^2\bar{\delta}\rho\right]\delta\mathbf{u} + \bar{\delta}e\delta(\rho\mathbf{u})\right\rangle\,, \tag{2}$$

where ε is the mean total energy injection rate (which is equal to the mean total energy dissipation rate), E is the total energy, $R = \rho\mathbf{u}\cdot\mathbf{u}'/2 + \rho e'$ and $\tilde{R} = \rho'\mathbf{u}'\cdot\mathbf{u}/2 + \rho' e$ are the relevant two-point correlation functions (defined at points M and M' separated by a vector $\mathbf{r}$), $\delta X \equiv X' - X'$, $\bar{\delta}X \equiv (X + X')/2$, C_s is the speed of sound, ρ is the density (the pressure is linked to the density by the isothermal relation $P = C_s^2\rho$) and $e = C_s^2\ln(\rho/\rho_0)$ is the internal energy. In comparison with the incompressible case [6], expression (2) reveals the presence of a new type of term (the two first terms in the right hand side) which is by nature compressible since it is proportional to the dilatation (*i.e.* the divergence of the velocity). This term has a major impact on the nature of compressible turbulence since it may act like a source or a sink for the mean energy transfer rate. We may simplify the previous equation by assuming isotropy; the exact relation can be written symbolically as:

$$-2\varepsilon = \mathcal{S}(r) + \frac{1}{r^2}\partial_r(r^2\mathcal{F}_r)\,, \tag{3}$$

where $\mathcal{S}$ is the source/sink term and $\mathcal{F}_r$ the radial component of the isotropic energy flux vector. If we define an effective mean total energy injection rate as $\varepsilon_{\text{eff}} \equiv \varepsilon + \mathcal{S}/2$, a simple interpretation of expression (3) can be proposed as we see in Fig. 1: whereas for a direct cascade the the energy flux vectors are oriented towards the center of the sphere, dilatation and compression are additional effects which act respectively in the opposite or in the same direction as the flux vectors (since $R - E$ is mainly negative).

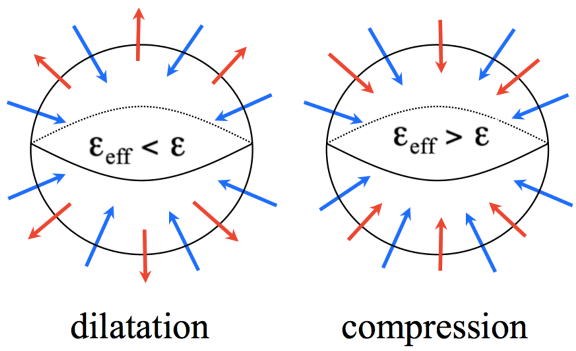

Figure 1. Dilatation (left) and compression (right) phases in space correlation for isotropic turbulence. In a direct cascade scenario the energy flux vectors (blue arrows) are oriented towards the center of the sphere. Dilatation and compression (red arrows) are additional effects which act respectively in the opposite or in the same direction as the flux vectors. From [4].

A power law spectrum prediction for compressible turbulence may be proposed by applying a dimensional analysis on equation (2). Dimensionally, we find $\varepsilon_{\text{eff}} r \sim \rho u^3$. By introducing the density-weighted fluid velocity, $\mathbf{v} \equiv \rho^{1/3}\mathbf{u}$, and following Kolmogorov we obtain $E^v(k) \sim \varepsilon_{\text{eff}}^{2/3} k^{-5/3}$, where $E^v(k)$ is the spectrum associated to the variable $\mathbf{v}$. Our prediction is compatible with the measurements recently made by direct numerical simulations [8] where the authors have noted that the exponent of the third-order velocity structure function is close to one if the field used is $\mathbf{v}$ instead of $\mathbf{u}$. As explained by several authors [7] in compressible turbulence we do not expect a constant flux in the inertial range. Here, the same conclusion is reached since we are dealing with an effective mean energy transfer rate. More precisely if we expect a power law dependence in k for the effective transfer rate [4] one arrives at the conclusion that a steeper power law spectrum may happen at the largest scales in $E^v(k) \sim k^{-19/9}$. The scale at which the transition happens between $-19/9$ and $-5/3$ may be the sonic scale $1/k_s$ as proposed in [2] where such power laws were detected; in our case, a rough estimate gives $k_s \sim \langle(\nabla \cdot \mathbf{u})/\delta\mathbf{u}\rangle$.

CONCLUSION

The present work opens important perspectives to further understand the nature of compressible turbulence in the asymptotic limit of large Reynolds numbers with the possibility to extend the analysis to magnetized fluids [1] with possibly other types of closures (*e.g.* polytropic gas), or to improve intermittency models by using the new relation – obtained by a statistical analysis at low order – as pivotal for a heuristic extension to statistical laws at higher order. We believe that astrophysics (*e.g.* interstellar turbulence) is one of the most important domain of application of the present work.

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