HIERARCHY OF STRUCTURE FUNCTION EQUATIONS FOR LOCALLY ISOTROPIC MHD TURBULENCE

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Abstract We investigate the structure of locally isotropic MHD turbulence by means of exact equations for two-point magnetic and velocity structure functions. To this end we make use of the calculus of isotropic tensors for MHD turbulence introduced by Chandrasekhar [1]. A hierarchy of structure function equations is obtained, beginning with the MHD analagon of Kolmogorov's four-fifths law of hydrodynamics. The next order equation relates the third- and fourth-order structure functions and is the first order which provides a direct dependence between the longitudinal and the transverse structure functions based on the dynamics. The obtained relations for two-dimensional MHD flows are checked by direct numerical simulations.

THE CALCULUS OF ISOTROPIC TENSORS IN MHD TURBULENCE

We consider the MHD equations in the following form

$$\frac{\partial}{\partial t}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{h}\cdot\nabla\mathbf{h} = -\frac{1}{\rho}\nabla\left(p+\frac{1}{2}\rho|\mathbf{h}|^2\right)+\nu\nabla^2\mathbf{u}, \tag{1}$$

$$\frac{\partial}{\partial t}\mathbf{h}+\mathbf{u}\cdot\nabla\mathbf{h}-\mathbf{h}\cdot\nabla\mathbf{u} = \lambda\nabla^2\mathbf{h}, \tag{2}$$

where

$$\mathbf{h}=\left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}}\mathbf{H}. \tag{3}$$

We introduce the velocity and magnetic increments

$$\mathbf{v}(\mathbf{x},\mathbf{x}',t)=\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{x}',t)=\mathbf{u}-\mathbf{u}' \quad\text{and}\quad \mathbf{b}(\mathbf{x},\mathbf{x}',t)=\mathbf{h}(\mathbf{x},t)-\mathbf{h}(\mathbf{x}',t)=\mathbf{h}-\mathbf{h}'. \tag{4}$$

and change to relative $\mathbf{r}=\mathbf{x}-\mathbf{x}'$ and center coordinates $\mathbf{X}=\frac{\mathbf{x}+\mathbf{x}'}{2}$.
In contrast to ordinary turbulence, in MHD we have to deal several times with quantities which are not invariant under the full rotation group. This is due to $\mathbf{h}$ being an axial vector which is unchanged under an reflexion, contrary to the true polar vector $\mathbf{u}$, which changes signs. Quantities which involve an odd number of magnetic increments like the cross helicity

$$\langle v_i b_j\rangle = D^{\mathbf{vb}}_{i\,j}(\mathbf{r},t)=D^{\mathbf{vb}}(r,t)\epsilon_{ijk}\frac{r_k}{r}, \tag{5}$$

thus lack the reflection symmetry, whereas quantities like $\langle v_i v_j\rangle$ or $\langle b_i b_j\rangle$ can be written in the usual form

$$\langle v_i v_j\rangle = D^{\mathbf{vv}}_{i\,j}(\mathbf{r},t)=\left(D^{\mathbf{vv}}_{r\,r}(r,t)-D^{\mathbf{vv}}_{t\,t}(r,t)\right)\frac{r_i r_j}{r^2}+D^{\mathbf{vv}}_{t\,t}(r,t)\delta_{ij}, \tag{6}$$

where $D^{\mathbf{vv}}_{r\,r}$ and $D^{\mathbf{vv}}_{t\,t}$ are the longitudinal and transverse structure functions.

SECOND ORDER STRUCTURE FUNCTIONS AND THE EQUATION OF ENERGY BALANCE IN MHD

The symmetry of the MHD equations provides the evolution equation of two structure functions of second order. The first one is the equation of energy balance in MHD

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left\langle v^2(\mathbf{r},t)+b^2(\mathbf{r},t)\right\rangle+\nabla_{\mathbf{r}}\cdot\left\langle \mathbf{v}(\mathbf{r},t)\frac{v^2(\mathbf{r},t)+b^2(\mathbf{r},t)}{2}\right\rangle-\nabla_{\mathbf{r}}\cdot\langle\mathbf{b}(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)\cdot\mathbf{b}(\mathbf{r},t)\rangle\\ =\;&\nu\nabla^2_{\mathbf{r}}\left\langle v^2(\mathbf{r},t)\right\rangle+\lambda\nabla^2_{\mathbf{r}}\left\langle b^2(\mathbf{r},t)\right\rangle-2\langle\varepsilon^{\mathbf{v}}(\mathbf{x})+\varepsilon^{\mathbf{b}}(\mathbf{x})\rangle,\end{aligned} \tag{7}$$

where $\varepsilon^{\mathbf{v}}(\mathbf{x})$ and $\varepsilon^{\mathbf{b}}(\mathbf{x})$ denote the corresponding local energy dissipation rates and where we have made use of the assumption of homogeneity. The generalization of the four-fifths law from hydrodynamic turbulence in the presence of a magnetic field can be derived from this equation according to

$$D^{\mathbf{vvv}}_{r\,r\,r}(r)-12C^{\mathbf{hhu}}_{t\,t\,r}(r)-\frac{24}{r^4}\int_0^r \mathrm{d}r'\,r'^3C^{\mathbf{uhh}}(r')=-\frac{4}{5}\langle\varepsilon^{\mathbf{v}}+\varepsilon^{\mathbf{b}}\rangle r. \tag{8}$$

This relation involves the correlation functions $\langle(h_j u_n-u_j h_n)h_i'\rangle = C^{\mathbf{uhh}}(r)\left(\frac{r_j}{r}\delta_{in}-\frac{r_n}{r}\delta_{ij}\right)$ and $\langle h_i h_j u_n'\rangle = C^{\mathbf{hhu}}_{i\,j\,n}(\mathbf{r})$. However, in the case of MHD turbulence this relation is not closed, since the source term from $C^{\mathbf{uhh}}(r)$ doesn't vanish in the inertial range. An approximation for the source term can be obtained from the other second order evolution equation of the cross helicity $\langle v_i b_j\rangle$ and allows one to draw analogies to the Iroshnikov-Kraichnan phenomenology.

NEXT ORDER STRUCTURE FUNCTION

In the next order of the hierarchy we have to deal the first time with structure functions containing the pressure gradient increment $P_i = \frac{1}{\rho}\frac{\partial}{\partial X_i}\left[p - p' + \frac{1}{2}\rho(|\mathbf{h}|^2 - |\mathbf{h}|'^2)\right]$. In the inertial range and under the assumption of homogeneity one gets

$$\begin{aligned} & \frac{\partial}{\partial t}\left\langle v_i v_j v_k + v_i b_j b_k + b_i v_j b_k + b_i b_j v_k \right\rangle + \frac{\partial}{\partial r_n}\left\langle v_i v_j v_k v_n - b_i b_j b_k b_n \right\rangle \\ & + \frac{\partial}{\partial r_n}\langle v_n (v_i b_j b_k + b_i v_j b_k + b_i b_j v_k)\rangle - \frac{\partial}{\partial r_n}\langle b_n (v_i v_j b_k + b_i v_j v_k + v_i b_j v_k)\rangle \\ = & -\langle (v_i v_j + b_i b_j) P_k + (v_i v_k + b_i b_k) P_j + (v_j v_k + b_j b_k) P_i \rangle \, . \end{aligned} \tag{9}$$

For stationary turbulence and in using the notation from above, we get

$$\frac{\partial}{\partial r_n}\left(D^{\mathbf{vvvv}}_{i\,j\,k\,n}(\mathbf{r}) - D^{\mathbf{bbbb}}_{i\,j\,k\,n}(\mathbf{r})\right) - \frac{\partial}{\partial r_n}\left(D^{\mathbf{vvbb}}_{i\,j,k\,n}(\mathbf{r}) + D^{\mathbf{vvbb}}_{k\,j,i\,n}(\mathbf{r}) + D^{\mathbf{vvbb}}_{i\,k,j\,n}(\mathbf{r})\right) = -T_{ijk}(\mathbf{r}), \tag{10}$$

where $T_{ijk}(\mathbf{r}, t)$ denotes the pressure contributions. Inserting the corresponding tensors from [6] yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(D^{\mathbf{vvvv}}_{r\,r\,r\,r}(r) - D^{\mathbf{bbbb}}_{r\,r\,r\,r}(r)\right)\right] - \frac{6}{r}\left(D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r) - D^{\mathbf{bbbb}}_{r\,r\,t\,t}(r) - D^{\mathbf{vvbb}}_{r\,r,t\,t}(r)\right) \quad = \quad -T_{rrr}(r), \tag{11}$$

for the longitudinal structure functions and

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r) - D^{\mathbf{bbbb}}_{r\,r\,t\,t}(r)\right)\right] - \frac{4}{3r}\left(D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r) - D^{\mathbf{bbbb}}_{t\,t\,t\,t}(r)\right) + \frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{r\,r,t\,t}(r) \quad = \quad -T_{rtt}(r), \tag{12}$$

for the mixed structure functions.
Similar equations were derived for hydrodynamic turbulence by Hill [2] and Yakhot [5]. However, the equations (12) and (13) show new features like cancellation effects for equipartition solutions and the influence of the magnetic pressure. The corresponding equations in two dimensions were derived and structure functions were evaluated from direct numerical simulations. The second and fourth order structure functions are depicted in Fig. 1 and reveal the interesting questions if rescaling relations between longitudinal and transverse structure functions similar to the ones presented in [4] can exist in MHD turbulence.

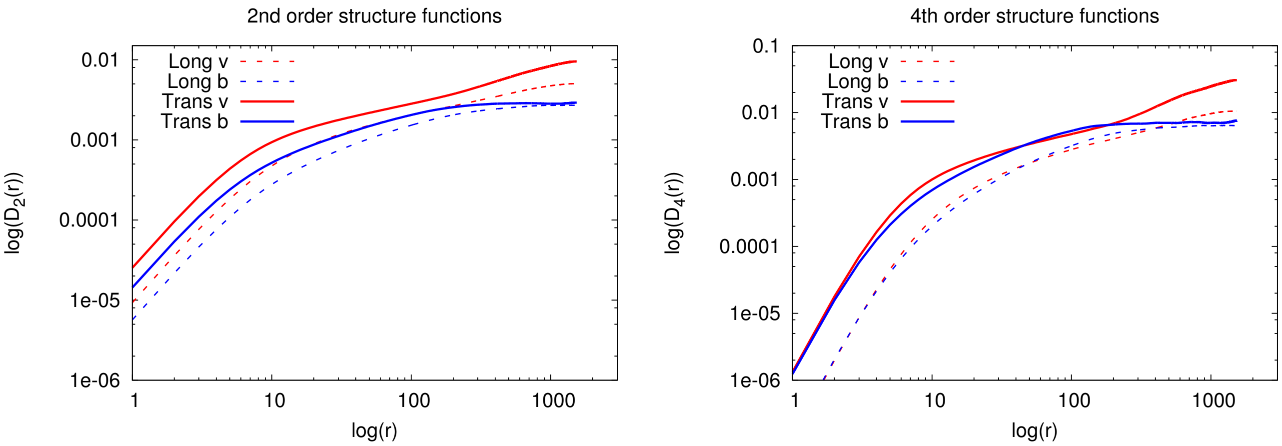

Figure 1. Structure functions of second and fourth order from DNS of the 2D MHD equations with hyperviscosity $\nu = \lambda = 2 \cdot 10^{-10}$ evaluated over 20 large-eddy turnover times.

References

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