THE FINITE-DIMENSIONAL REPRESENTATIONS OF DIFFERENTIATIONS IN FUNCTIONAL RINGS AND INTEGRABILITY OF A RIEMANN TYPE HYDRODYNAMIC HIERARCHY

Anatolij K. Prykarpatski[1], Denis Blackmore [2]

[1] *The Department of Applied Mathematics at AGH University of Science and Technology, Cracow 30059, Poland*

[2]*Department of Mathematical Sciences at the New Jersey Institute of Technology (NJIT), Newark NJ 07102, USA*

Abstract A differential-algebraic approach [1, 2] to studying the Lax type integrability of an infinite hierarchy of generalized Riemann type hydrodynamic systems is developed. The related bi-Hamiltonian integrability and compatible Poisson structures are analyzed by means of the symplectic and gradient-holonomic methods.

DIFFERENTIAL-ALGEBRAIC SETTING

We consider the ring $\mathcal{K} := \mathbf{R}\{\{x,t\}\}$, $(x,t) \in \mathbf{R}^2$, of convergent germs of real-valued smooth Schwartz type functions from $S(\mathbf{R}^2;\mathbf{R})$ and construct the associated differential quotient ring $\mathcal{K}\{u\} := Quot(\mathcal{K}[\Theta u])$ with respect to a functional variable $u \in \mathcal{K}$. Here Θ denotes the standard monoid of all commuting differentiations $D_x := \partial/\partial x$ and $\partial/\partial t$, satisfying the standard Leibnitz condition, and defined by the natural relationships $D_x(x) = 1 = \partial t/\partial t$, $\partial x/\partial t = 0 = D_x(t)$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential if the condition $I\{u\} = \Theta I\{u\}$ holds. In the differential ring $\mathcal{K}\{u\}$, one can introduce the next two naturally defined differentiations $D_t, D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, satisfying the Lie algebraic commuting relationship $[D_x,\ D_t] = u_x D_x$. For an arbitrarily chosen function $u \in \mathcal{K}$, the only its representation in the ideal $\mathcal{K}\{u\}$ is of the form $D_t = \partial/\partial t + u\partial/\partial x; D_x = \partial/\partial x$. Nonetheless, if some additional nonlinear differential constraints $Z[u, D_x u, D_t u, ...] = 0$ are imposed on the function $u \in \mathcal{K}$, other nontrivial differentiations in the corresponding reduced ideal $\overline{\mathcal{K}}\{u\}$, related with some specially extended invariant differential ideals $\mathcal{I}\{u\} \subset \overline{\mathcal{K}}\{u\}$ can exist. This situation is analyzed in detail and the corresponding, polynomially dependent on $u \in \mathcal{K}$ and its derivatives with respect to the differentiation D_x, representation of the above Lie algebraic relationship is constructed.

THE LAX TYPE REPRESENTATION

The following differential constraint

$$D_t^{N-1} u = v, \qquad D_t v = \bar{z}_x^s, \qquad D_t \bar{z} = 0, \tag{1}$$

where $N, s \in \mathbf{N}$, being imposed on the ideal $\mathcal{K}\{u\}$, determines an infinite hierarchy of the Riemann type hydrodynamic systems. We have proved the following proposition.

Proposition. The infinite hierarchy (1) of the Riemann type hydrodynamical systems for $N, s \in \mathbf{N}$ is Lax type integrable and bi-Hamiltonian on the functional manifold M. In the case $N = 2 = s$ the flow (1) is a bi-Hamiltonian system with respect to two compatible Poissonian structures $\vartheta, \eta : T^*(M) \to T(M)$

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2} D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} z_x - 2z \\ 0 & z_x \partial^{-1} + 2z & 0 \end{pmatrix}, \tag{2}$$

possessing an infinite hierarchy of mutually commuting conservation laws and a non-autonomous Lax representation of the form

$$D_t f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix} f, D_x f = \begin{pmatrix} \begin{matrix}\lambda^2 u\sqrt{z} \\ -\lambda^3 t u \sqrt{z} \\ \lambda^4 (tuv - u^2) - \\ -\lambda^2 u_x/\sqrt{z}\end{matrix} & \begin{matrix}\lambda v \sqrt{z} \\ -\lambda^2 t v \sqrt{z} \\ -\lambda v_x/\sqrt{z} + \\ +\lambda^3 (tv^2 - uv)\end{matrix} & \begin{matrix} z \\ -\lambda t z \\ \lambda^2 \sqrt{z}(u - tv) - \\ -z_x/2z\end{matrix} \end{pmatrix} f, \tag{3}$$

where $\lambda \in \mathbf{R}$ is an arbitrary spectral parameter and $f \in \mathcal{K}$.

References

[1] A.K. Prykarpatsky, O.D. Artemovych, Z. Popowicz and M.V. Pavlov. Differential-algebraic integrability analysis of the generalized Riemann type and Korteweg–de Vries hydrodynamical equations. J. Phys. A: Math. Theor. 43, 295205 (13pp), 2010

[2] Y.A. Prykarpatsky, O.D. Artemovych, M.V. Pavlov, and A.K. Prykarpatsky. Differential-algebraic and bi-Hamiltonian integrability analysis of the Riemann hierarchy revisited. J. Math. Phys. 53, 103521, 2012