PROGRESS ON EDDY-VISCOSITY MODELS FOR LES: NEW DIFFERENTIAL OPERATORS AND DISCRETIZATION METHODS

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Abstract The incompressible Navier-Stokes equations constitute an excellent mathematical modelization of turbulence. Unfortunately, attempts at performing direct simulations are limited to relatively low-Reynolds numbers. Therefore, dynamically less complex mathematical formulations are necessary for coarse-grain simulations. Eddy-viscosity models for Large-Eddy Simulation (LES) is an example thereof: they rely on differential operators that should be able to capture well different flow configurations (laminar and 2D flows, near-wall behavior, transitional regime...). In the present work, several differential operators are derived from the criterion that vortex-stretching mechanism must stop at the smallest grid scale. Moreover, since the discretization errors may play an important role a novel approach to discretize the viscous term with spatially varying eddy-viscosity is used. It is based on basic operators; therefore, the implementation is straightforward even for staggered formulations.

INTRODUCTION

We consider the simulation of the incompressible Navier-Stokes (NS) equations. In primitive variables they read

$$\partial_t \boldsymbol{u} + \mathcal{C}(\boldsymbol{u},\boldsymbol{u}) = \mathcal{D}\boldsymbol{u} - \nabla p, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u}$ denotes the velocity field, p represents the pressure, the non-linear convective term is given by $\mathcal{C}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}$, and the diffusive term reads $\mathcal{D}\boldsymbol{u} = \nu\Delta\boldsymbol{u}$, where ν is the kinematic viscosity. Direct simulations at high Reynolds numbers are not feasible because the convective term produces many scales of motion. Hence, in the foreseeable future numerical simulations of turbulent flows will have to resort to models of the small scales. The most popular example thereof is the Large-Eddy Simulation (LES). Shortly, LES equations result from filtering the NS equations in space

$$\partial_t \overline{\boldsymbol{u}} + \mathcal{C}(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}}) = \mathcal{D}\overline{\boldsymbol{u}} - \nabla\overline{p} - \nabla\cdot\tau(\overline{\boldsymbol{u}})\,; \qquad \nabla\cdot\overline{\boldsymbol{u}} = 0, \tag{2}$$

where $\overline{\boldsymbol{u}}$ is the filtered velocity and $\tau(\overline{\boldsymbol{u}})$ is the subgrid stress tensor and approximates the effect of the under-resolved scales, *i.e.* $\tau(\overline{\boldsymbol{u}}) \approx \overline{\boldsymbol{u}\otimes\boldsymbol{u}} - \overline{\boldsymbol{u}}\otimes\overline{\boldsymbol{u}}$. Then, the closure problem consists on replacing (approximating) the tensor $\overline{\boldsymbol{u}\otimes\boldsymbol{u}}$ with a tensor depending only on $\overline{\boldsymbol{u}}$ (and not $\boldsymbol{u}$). Because of its inherent simplicity and robustness, the eddy-viscosity assumption is by far the most used closure model

$$\tau(\overline{\boldsymbol{u}}) \approx -2\nu_e S(\overline{\boldsymbol{u}}), \tag{3}$$

where ν_e denotes the eddy-viscosity and $\tau(\overline{\boldsymbol{u}})$ is considered traceless without the loss of generality.

RESTRAINING THE PRODUCTION OF SMALL SCALES

The essence of turbulence are the smallest scales of motion. They result from a subtle balance between convective transport and diffusive dissipation. Numerically, if the grid is not fine enough, this balance needs to be restored by a turbulence model. Hence, the success of a turbulence model depends on the ability to capture well this (im)balance. Let us consider an arbitrary part of the domain flow, Ω, with periodic boundary conditions. The inner product is defined in the usual way: $(a,b) = \int_\Omega a\cdot b d\Omega$. Then, taking the L^2 inner product of (1) with $-\Delta\boldsymbol{u}$ leads to the enstrophy equation

$$1/2\|\boldsymbol{\omega}\|_t^2 = (\boldsymbol{\omega},\mathcal{C}(\boldsymbol{\omega},\boldsymbol{u})) - \nu(\nabla\boldsymbol{\omega},\nabla\boldsymbol{\omega}), \tag{4}$$

where $\|\boldsymbol{\omega}\|^2 = (\boldsymbol{\omega},\boldsymbol{\omega})$ and the convective term contribution $(\mathcal{C}(\boldsymbol{u},\boldsymbol{\omega}),\boldsymbol{\omega}) = 0$ vanishes because of the skew-symmetry of $\mathcal{C}$. Following [3], the vortex-stretching term can be expressed in terms of the invariant $R = -1/3tr(S^3) = -det(S)$

$$(\boldsymbol{\omega},\mathcal{C}(\boldsymbol{\omega},\boldsymbol{u})) = -\frac{4}{3}\int_\Omega tr(S^3)d\Omega = 4\int_\Omega R d\Omega = 4\widetilde{R}, \tag{5}$$

whereas the diffusive terms may be bounded in terms of the invariant $Q = -1/2tr(S^2)$

$$(\nabla\boldsymbol{\omega},\nabla\boldsymbol{\omega}) = -(\boldsymbol{\omega},\Delta\boldsymbol{\omega}) \le -\lambda_\Delta(\boldsymbol{\omega},\boldsymbol{\omega}) = 4\lambda_\Delta\int_\Omega Q d\Omega = 4\lambda_\Delta\widetilde{Q}, \tag{6}$$

where $\lambda_\Delta < 0$ is the largest (smallest in absolute value) non-zero eigenvalue of the Laplacian operator Δ on Ω and $\widetilde{(\cdot)}$ denotes the integral over Ω. However, it relies on the accurate estimation of λ_Δ on Ω. The latter may be cumbersome, especially on unstructured grids. Alternatively, it may be (numerically) computed directly from $(\nabla\boldsymbol{\omega}, \nabla\boldsymbol{\omega})$ or, even easier, by simply noticing that $(\nabla\boldsymbol{\omega}, \nabla\boldsymbol{\omega}) = 4\int_\Omega Q(\boldsymbol{\omega}) d\Omega = 4\widetilde{Q(\boldsymbol{\omega})}$. However, from a numerical point-of-view, these integrations are not straightforward. Instead, recalling that $\nabla \times \nabla \times \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \Delta \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u} = 0$, a more appropriate expression can be obtained as follows

$$(\nabla\boldsymbol{\omega}, \nabla\boldsymbol{\omega}) = -(\boldsymbol{\omega}, \Delta\boldsymbol{\omega}) = (\boldsymbol{\omega}, \nabla \times \nabla \times \boldsymbol{\omega}) = (\nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\omega}) = (\Delta\boldsymbol{u}, \Delta\boldsymbol{u}) = \|\Delta\boldsymbol{u}\|^2. \quad (7)$$

Then, to prevent a local intensification of vorticity, *i.e.* $\|\boldsymbol{\omega}\|_t \leq 0$, the inequality $H_\Omega \leq \nu(\Delta\boldsymbol{u}, \Delta\boldsymbol{u})/(\boldsymbol{\omega}, S\boldsymbol{\omega})$ must be satisfied, where H_Ω denotes the overall damping introduced by the model in the (small) part of the domain Ω. Additionally, the dynamics of the large scales should not be significantly affected by the (small) scales contained within the domain Ω, *i.e.* $(\boldsymbol{\omega}, S\boldsymbol{\omega}) < 0$. Then, from Eq.(5) and noticing that $0 < H_\Omega \leq 1$, a proper definition of the overall damping factor follows $H_\Omega = \min\left\{\nu\|\Delta\boldsymbol{u}\|^2/|\widetilde{R}|, 1\right\}$. An eddy-viscosity model, $\tau(\overline{\boldsymbol{u}}) = -2\nu_e S(\overline{\boldsymbol{u}})$, adds the dissipation term $(\nabla\overline{\boldsymbol{\omega}}, \nu_e \nabla\overline{\boldsymbol{\omega}})$ to the enstrophy equation. In this case, the eddy-viscosity, ν_e, results

$$\nu_e = \max\left\{(4|\widetilde{R}| - \nu\|\Delta\overline{\boldsymbol{u}}\|^2)/\|\Delta\overline{\boldsymbol{u}}\|^2, 0\right\}. \quad (8)$$

This analysis can be extended further for other differential operators. For instance, $\tau'(\overline{\boldsymbol{u}}) = 2\nu_e' S(\Delta\overline{\boldsymbol{u}})$ and $\tau''(\overline{\boldsymbol{u}}) = -2\nu_e'' S(\Delta^2\overline{\boldsymbol{u}})$, where $\Delta^2 \equiv \Delta\Delta$ is the bi-Laplacian, lead to the following hyperviscosity terms in the enstrophy equation $-(\nabla\overline{\boldsymbol{\omega}}, \nu_e' \nabla\Delta\overline{\boldsymbol{\omega}})$ and $(\nabla\overline{\boldsymbol{\omega}}, \nu_e'' \nabla\Delta^2\overline{\boldsymbol{\omega}})$. Then, following similar reasonings, ν_e' and ν_e'' follow

$$\nu_e' = \max\left\{-(4|\widetilde{R}| - \nu\|\Delta\overline{\boldsymbol{u}}\|^2)/(\Delta\overline{\boldsymbol{u}}, \Delta^2\overline{\boldsymbol{u}}), 0\right\} \quad \text{and} \quad \nu_e'' = \max\left\{(4|\widetilde{R}| - \nu\|\Delta\overline{\boldsymbol{u}}\|^2)/\|\Delta^2\overline{\boldsymbol{u}}\|^2, 0\right\}. \quad (9)$$

DISCRETIZING THE VISCOUS TERM WITH SPATIALLY VARYING EDDY-VISCOSITY

The NS equations (1) with constant physical properties are discretized on a staggered grid using a fourth-order symmetry-preserving discretization [4]. Here we propose to apply the same ideas to discretize the eddy-viscosity model (3) for LES (2). To obtain the Eq.(1) (with ν replaced by $\nu + \nu_e$) from Eqs.(2)-(3) with constant ν_e notice that $2\nabla \cdot S(\boldsymbol{u}) = \nabla \cdot \nabla\boldsymbol{u} + \nabla \cdot (\nabla\boldsymbol{u})^T$ and recall the vector calculus identity $\nabla \cdot (\nabla\boldsymbol{u})^T = \nabla(\nabla \cdot \boldsymbol{u})$ to cancel out the second term. However, for non-constant ν_e, the discretization of $\nabla \cdot (\nu_e(\nabla\boldsymbol{u})^T)$ needs to be addressed. This can be quite cumbersome especially for staggered formulations. The standard approach consist on discretizing the term $\nabla \cdot (\nu_e(\nabla\boldsymbol{u})^T)$ directly. However, this implies many *ad hoc* interpolations that tends to smear the eddy-viscosity, ν_e. This may (negatively?) influence the performance of eddy-viscosity especially near the walls. Instead, an alternative form was proposed in [2]. Shortly, with the help of vector calculus it can be shown that $\nabla \cdot (\nu_e(\nabla\boldsymbol{u})^T) = \nabla(\nabla \cdot (\nu_e\boldsymbol{u})) - \nabla \cdot (\boldsymbol{u} \otimes \nabla\nu_e)$. Then, recalling that the flow is incompressible, the second term in the right-hand-side can be written as $\nabla \cdot (\boldsymbol{u} \otimes \nabla\nu_e) = (\boldsymbol{u} \cdot \nabla)\nabla\nu_e = \mathcal{C}(\boldsymbol{u}, \nabla\nu_e)$, *i.e.*

$$\nabla \cdot (\nu_e(\nabla\boldsymbol{u})^T) = \nabla(\nabla \cdot (\nu_e\boldsymbol{u})) - \mathcal{C}(\boldsymbol{u}, \nabla\nu_e). \quad (10)$$

In this way, consistent approximations of Eqs.(2)-(3) can be constructed without introducing new interpolation operators.

CONCLUDING REMARKS AND FUTURE RESEARCH

In the context of LES, three eddy-viscosity-type models have been obtained. Namely, (i) $\tau(\overline{\boldsymbol{u}}) = -2\nu_e S(\overline{\boldsymbol{u}})$, (ii) $\tau'(\overline{\boldsymbol{u}}) = 2\nu_e' S(\Delta\overline{\boldsymbol{u}})$ and (iii) $\tau''(\overline{\boldsymbol{u}}) = -2\nu_e'' S(\Delta^2\overline{\boldsymbol{u}})$, where ν_e, ν_e' and ν_e'' are given by Eqs.(8) and (9), respectively. Notice that, apart from R, these models can be straightforwardly implemented by re-using the discrete diffusive operator. They can be related with already existing approaches. Firstly, the model (i) is almost the same than the recently proposed QR-model [3]. Essentially, they only differ on the calculation of the diffusive contribution to the enstrophy equation: instead of making use of the equality (7) it is bounded by means of the inequality (6), therefore, the eddy-viscosity is given by $\nu_e \propto \lambda_\Delta^{-1}|\widetilde{R}|/\widetilde{Q}$ instead of Eq.(8). Regarding the models (ii) and (iii) they can be respectively related to the well-known small-large and small-small variational multiscale methods [1] by noticing that $\boldsymbol{u}' = -(\epsilon^2/24)\Delta\boldsymbol{u} + \mathcal{O}(\epsilon^4)$. All these models switch off ($R \to 0$) for laminar (no vortex-stretching), 2D flows ($\lambda_2 = 0 \to R = 0$) and near the wall ($R \propto y^1$). To test the performance of these new turbulence models in conjunction with the new discretization approach is part of our research plans. In particular, we plan to test them for a turbulent channel flow and square duct at $Re_\tau = 1200$.

References

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