MOTION OF TOROIDAL BUNDLES OF VORTEX RINGS

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Abstract Using the thin filament approximation, we study the motion of toroidal bundles of inviscid vortex rings, and compare results with experiments in superfluid helium.

INTRODUCTION

This work is motivated by experiments [1] in which macroscopic vortex rings of diameter $D \approx 1$ cm were generated in superfluid helium by the usual piston-cylinder arrangement, and were observed to propagate at constant velocity for a distance of approximately 10 diameters. The circulation Γ about these large vortex rings, measured acoustically, was found to be of the order of $1\ \mathrm{cm}^2/\mathrm{s}$. The natural interpretation [1] of the experiments is that toroidal bundles, consisting of over one thousand concentric vortex rings, each ring carrying one quantum of circulation κ, travelled in a coherent way for a significant distance without disintegrating.
It is well known that in superfluid helium a single vortex ring of radius R is stable and travels with speed

$$v = \frac{\kappa}{4\pi R}[\ln(8R/a_0) - 1/2], \tag{1}$$

where $a_0 \approx 10^{-8}$ cm is the vortex core radius and $\kappa = h/m = 9.97 \times 10^{-3}\ \mathrm{cm}^2/\mathrm{s}$, where h is Planck's constant and m is the mass of one helium atom. The question which we address is whether a large number N of such coaxial vortex rings, forming a toroidal vortex bundle, can travel together coherently for sufficiently long distances.
To find the answer, we perform three-dimensional numerical simulations of the motion of bundles of vortex rings. An example of a vortex bundle is shown in Fig. 1. Our model is based on the thin filament approximation. The vortex filaments are descrived as space curves $\mathbf{s}(t)$ (where t is time) which move according to the Biot-Savart law

$$\frac{d\mathbf{s}}{dt} = -\frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{(\mathbf{s}-\mathbf{r})}{|\mathbf{s}-\mathbf{r}|^3} \times \mathbf{dr}, \tag{2}$$

where the integral extends along all vortices. The numerical algorithm to de-singularize and integrate Eq. 2 in time is described elsewhere [2]. The initial condition consists of N circular rings of average radius R, with circulation oriented in the same direction and hexagonal pattern on the (initially circular) cross section of radius a.

RESULTS

We study the evolution of vortex bundles as a function of N, with experimentally realistic values of R and a, and determine their translational speed. The computational cost of the Biot-Savart law limits our calculations to $N < 100$, but still provides us with insight into the experiment.
Our numerical simulations show that vortex bundles are relatively stable, and remain coherent over the observed distance ($\approx 10D$), after which we stop the calculation. The individual vortex rings which form a bundle, being oriented in the same direction, rotate about each other, essentially performing a generalised leapfrogging motion. The cross section of the bundle, initially circular, becomes slightly flattened.
We find that for sufficiently small values of R/a the vortex rings develop Kelvin waves. In some cases the amplitude of these waves becomes larger than the separation of the vortex filaments, and vortex reconnections take place (our code allows vortex reconnections [3] to model superfluid helium more realistically). The resulting turbulent vortex bundles, illustrated in Fig. 2, still remains coherent and travel a distance consistent with the observations.

References

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Figure 1. Example of laminar bundle of vortex rings (note that the thickness of each constituent vortex ring is for the sake of visualization only, as the radius of the vortex core is infinitesimal).

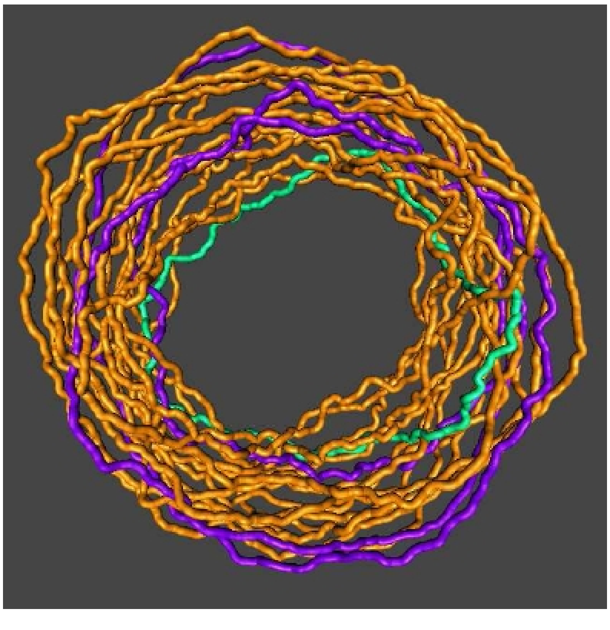

Figure 2. Example of turbulent bundle of vortex rings. Note the Kelvin waves on each constituent vortex ring.